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The onset of drag reduction by dilute polymer additives, and the maximum drag reduction asymptote

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Drag reduction due to dilute addition of high polymers has been known for about fifty years. In spite of this long history, many aspects of the problem remain poorly understood. Two of its features for pipe flow are considered here in the context of the elastic theory of de Gennes: the onset of drag reduction and the so-called maximum drag reduction (MDR) asymptote. A cautious conclusion is that a version of the theory agrees qualitatively with existing experiments.

1. Introduction

The addition of small amounts of long-chain polymers to flowing fluids produces large effects on a wide range of phenomena such as the stability of laminar motion, transition to turbulence, vortex formation and break-up, turbulent transport of heat, mass and momentum, and surface pressure fluctuations. Other additives such as surfactants have qualitatively similar effects. The literature on these studies is voluminous, and the comprehensive bibliography prepared by Nadolink & Haigh (1995) lists some 2500 papers. The feature that has attracted the most attention is the reduction of friction drag by up to 80% when only a few tens of parts per million by weight of long-chain polymers are added to a working fluid such as water. The reviews by Lumley (1969), Liaw, Zakin & Patterson (1971), Hoyt (1972), Landhal (1973), Virk (1975), and McComb (1990) provide some perspective on the problem. Four recent experimental papers on the subject, chosen as representative though with no special discernment, are by Walker & Tiederman (1990), Tong, Goldburg & Huang (1992), Groisman & Steinberg (1996), and Cadot, Bonn & Douady (1998); two papers with advances arising from direct numerical simulations are those of den Toonder *et al.* (1997) and Dimitropoulos, Sureshkumar & Beris (1998).

It is fair to say that the extensive – and continuing – activity has not yet produced a firm grasp of the mechanisms of drag reduction; even the most recent papers on the subject assess the overall situation in similar terms. This incomplete understanding is not surprising because the problem combines the interaction of two fields, namely polymers in solution and turbulent flows, neither of which is completely understood. The difficulty is compounded by the fact that most of the early experimental work – allowing for notable exceptions – was motivated by applied considerations, less by the fundamentals of the phenomenon. For example, very little work exists in which polymer properties were characterized and controlled adequately. However, there are reasons for renewed optimism. In addition to the sophisticated simulations of the kind

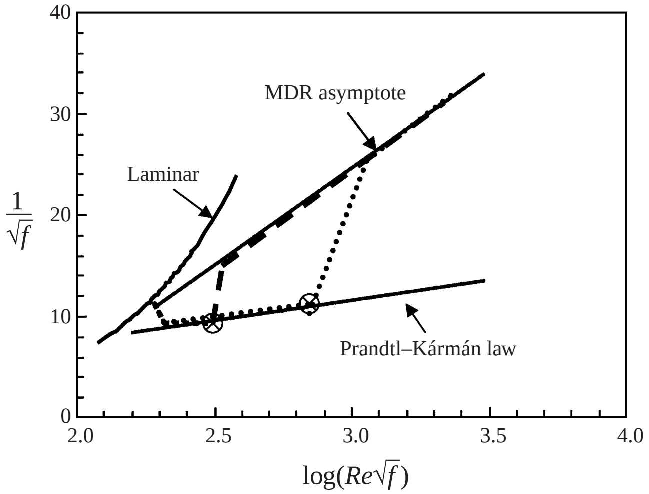

Figure 1. A schematic illustrating the onset of drag reduction and the maximum drag-reduction asymptote. The coordinates are those used in standard plots of pressure drop in pipe flows. The so-called Prandtl–Kármán law corresponds to the turbulent flow of Newtonian fluids. The dotted and dashed lines represent qualitatively the friction factor behaviour when a small amount of polymer is mixed with the fluid. The dashed line is for a larger polymer concentration. The line marked MDR asymptote represents the empirically observed limit on polymer drag reduction (Virk 1975).

already mentioned, our understanding of polymer solutions has improved significantly over the last two decades, both at the molecular level and in terms of constitutive equations. The theoretical ideas developed during the sixties and seventies have not yet been utilized fully, and none has been tested against experiments quantitatively. This provides the motivation for re-examining this old problem.

A plausible theory can be tested against many aspects of drag reduction in pipes: qualitative and quantitative changes in the mean velocity profile, alteration of turbulent stresses and the reduction of the correlation between the streamwise and radial velocity fluctuations, reduction of high-order moments of fluctuations, modification of the turbulent structure, especially at the wall, and so forth. Here, we shall restrict attention to two first-order effects, both of which are explained best with reference to figure 1. The figure shows the conventional friction drag for pipe flows in the so-called Prandtl–Kármán (P-K) coordinates. The friction factor, f, is related to the pressure drop Δp across a length L of the pipe of radius R as

$$f = \frac{\Delta p}{\rho V^2}\frac{R}{L}. \tag{1}$$

Here, ρ is the density of the fluid and V is the section-average mean velocity in the pipe. The Reynolds number $Re = 2VR/\nu$, ν being the kinematic viscosity of the solution. In these coordinates, the pressure drop in the turbulent pipe flow of a Newtonian fluid follows the straight line denoted as the Prandtl–Kármán law. A dilute solution of a polymer follows the P-K law up to a certain Reynolds number but abruptly departs from it thereafter. This point of departure is the onset of drag reduction. The onset occurs at a lower Reynolds number if the polymer concentration is increased, as shown qualitatively in the figure.

As one increases the Reynolds number, the drag reduction curves of figure 1 will ultimately merge with the line denoted as the 'MDR asymptote'. It has been noted by Virk (1975) that increasing the polymer concentration does not lead to drag reductions in excess of this asymptote – which is strikingly insensitive to polymer species, molecular weight, or the polymer–solvent pair.† A satisfactory explanation for the existence of a unique bounding mechanism for drag reduction is still lacking.

In summary, then, the present objective is to understand better the onset of drag reduction and the MDR asymptote. For this purpose, we shall invoke a theory based on the elastic behaviour of stretched polymers (Tabor & de Gennes 1986; de Gennes 1990). While some details here differ from de Gennes', his theory will be applied unaltered in most major respects. The rest of this paper is organized as follows. Section 2 makes a few general remarks on why the elastic theory deserves our consideration. In § 3, we shall recapitulate the essential elements of that theory, and present in § 4 some empirical evidence to show that the theory is qualitatively correct for predicting the onset of drag reduction. This success has encouraged us to consider the MDR asymptote as well (§ 5). The paper ends with a summary of conclusions in § 6.

2. Why consider the elastic theory?

An important basis for explaining drag reduction is the so-called 'time criterion' (e.g. Hershey & Zakin 1967; Lumley 1969). For convenience, the criterion may be split into several parts. First, for the turbulence to have any effect on the polymer, the characteristic time scale of the former should be smaller than the appropriate relaxation time of the latter. Second, whenever this condition is satisfied, the polymer molecules become substantially stretched. This is called the coil-stretch transition. In this extended state, the elongational viscosity increases by a factor of the order of ten thousand (e.g. Metzner & Metzner 1970; Landhal 1973; Hinch 1977). Third, this phenomenal increase in elongational viscosity manifests preferentially near the wall because the extensional strain rates are the highest there. Finally, it is supposed that the increased elongational viscosity suppresses turbulent fluctuations, increases the buffer layer thickness and reduces wall friction. Lumley (1973), for instance, produced detailed arguments to show that these considerations are plausible (see, also, Virk *et al.* 1967 and Landhal 1973).

The connection between the *fluctuating* strain rates and the large extensional viscosity, circumstantial to a large extent until recently, is being understood better now. The most fruitful attack comes from direct numerical simulations using model equations for the polymer. Sureshkumar, Beris & Handler (1997) and Dimitropoulos *et al.* (1998) have simulated turbulent flows of a polymer solution using constitutive equations derived from kinetic and network theories. Their calculations suggest that the enhanced extensional viscosity plays a large role in drag reduction. However, in order to match the polymer effects at the low Reynolds number of the simulations, these authors were forced to introduce greater elasticity in their model than is appropriate to drag-reducing polymers. Thus, the simulations are still some way away from providing a complete picture.

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† One can surpass the MDR asymptote by using surfactants instead of polymers, but the drag reduction in this case is due to the formation of networks of rod-like micelles. This mechanism is different from that involving polymers (Zakin, Myska & Chara 1996; Povkh, Stupin & Aslanov 1998; van Doorn, White & Sreenivasan 1998), and will not be considered here. Further, for each combination of the polymer, its concentration and flow characteristics, the data lying on the MDR line eventually 'peel off' if the Reynolds number exceeds some large value. This issue will not be considered here either. A few possible views on MDR will be mentioned in § 5.

Other explanations for drag reduction have always existed. Metzner & Park (1964), for instance, noted that non-Newtonian properties of a purely viscous nature produce no drag reduction, and drew attention to the importance of viscoelasticity. Joseph (1990) noted that the primary effect is likely to be elastic because the polymers always attenuate turbulence at small scales (e.g. Harder & Tiederman 1991), suggesting the existence of a natural cut-off scale provided by the elastic wave speed. In particular, Tabor & de Gennes (1986) and de Gennes (1990) thought that the coil-stretch transition does not occur in turbulent flows with randomly fluctuating strain rates, and that, if moderately stretched, the polymers produce no measurable change in viscosity. In their view, the major effect arises only when the elastic energy stored by the partially stretched polymers becomes comparable to the turbulent energy. The elastic energy increases with decreasing length scale of turbulence (because of increased stretching characteristic of turbulence dynamics), whereas the turbulent energy decreases with the scale size. When the two energies become comparable, the elastic energy interferes with the usual turbulent cascade mechanism (e.g. Monin & Yaglom 1975) without allowing it to proceed all the way to the Kolmogorov scale. De Gennes (1990) added some details but stopped short of fully elucidating the connection of this effect to drag reduction. The essential idea, however, is that the nonlinear action that generates small scales of turbulence will be terminated at some scale larger than the Kolmogorov scale. This larger length scale can lead to increased buffer layer thickness and reduced drag.

Since the elastic theory does not necessarily concern itself with the wall region of the pipe, it is often thought that the best support for it comes from qualitative experiments in which a reduction of wall stress begins to occur well before a polymer, injected into the core of the pipe flow of Newtonian fluids, reaches the wall. De Gennes (1990) himself (see also de Gennes & Badoz 1996) invoked the experiments of McComb & Rabie (1979) and of Bewersdorff *et al.* (1993) to emphasize this view. However, it is unclear if the experiments show this feature clearly enough (McComb 1990); further, recent experiments (e.g. Cadot *et al.* 1998 and our own, unpublished) on the injection of polymer solutions into water flows, do not provide unequivocal support for the view that significant drag reduction begins before the polymer reaches the wall.

We wish to emphasize that the lack of clear evidence on this front does not in itself negate the essentials of the elastic theory. In particular, two observations have driven us to consider that theory further. First, careful measurements in the late 1980s have indeed shown that the onset of drag reduction depends systematically on polymer concentration (e.g. Nadolink 1987) – an aspect that is not evident in the time criterion. While extensional models can indirectly exhibit the concentration dependence with respect to the onset of drag reduction, the elastic theory provides the cleanest, and the most direct, means of demonstrating the existence of a threshold concentration required for drag reduction. It would therefore appear that the elastic theory is worth testing against experiment. Second, recent experiments (Smith & Chu 1998) have suggested that partial stretching of polymer molecules is perhaps the rule even under rapid strain rates, with the details of stretching depending on the initial conditions of the polymer. This observation underlines the merit of one of the premises of the elastic theory. We shall therefore directly test quantitative predictions of the theory for the case of premixed polymer solutions flowing through pipes. The evidence to be presented suggests that the theory may have the right ingredients in it. It is to the elucidation of this evidence that the rest of the paper is devoted.

For completeness, two other comments should be made. A comprehensive simulation of the flow of polymeric fluids (den Toonder *et al.* 1998) suggests that the anisotropy of the stress caused by the extension of polymeric coils, rather than viscoelasticity, may be the key to drag reduction. However, the model used in the simulations does not have an elastic modulus for the shear, and so cannot fully address elastic effects. Using a version of the elastic theory, though from a different perspective from the present, Bhattacharjee & Thirumalai (1991) have shown that the onset of drag reduction indeed requires a threshold concentration of the polymer.

3. The elastic theory of drag reduction

3.1. *Homogeneous turbulence*

Let us first consider turbulent fluctuations at high Reynolds numbers, away from the wall. For this case, the assumption of homogeneity is reasonable and the classical picture of Kolmogorov (1941) is roughly valid. We shall ignore the effect of spatial intermittency of small-scale turbulence (Kolmogorov 1962; Frisch 1995; Sreenivasan & Antonia 1997), but this can be incorporated without much difficulty. For any length scale r in the inertial range (the intermediate range roughly between the large scale and the dissipative scale of turbulence), there corresponds a time scale τ_r given by the relation (Monin & Yaglom 1975)

$$\tau_r = (r^2/\langle\varepsilon\rangle)^{1/3}, \tag{2}$$

where $\langle\varepsilon\rangle$ is the average dissipation rate of the turbulence kinetic energy. If one substitutes for r the Kolmogorov scale $\eta \equiv (\nu^3/\langle\varepsilon\rangle)^{1/4}$, one obtains the estimate appropriate to the dissipation scale, equal to $(\nu/\langle\varepsilon\rangle)^{1/2}$. This is the smallest characteristic time scale in turbulence.

A convenient measure of the relaxation time of a linear chain polymer is that introduced by Zimm (1956); this is the average time taken by a stretched coil to return to its 'equilibrium' configuration as a result of Brownian bombardment. The Zimm relaxation time varies with the coil size, represented by its radius of gyration at rest, R_G. For a polymer in good solvent, the relaxation time is given by Flory's (1971) relation

$$\frac{1}{T_z} = \frac{\kappa T}{\eta_o R_G^3}, \tag{3}$$

where κ is the Boltzmann constant, T is the solution temperature, and η_o is the solvent viscosity. For flexible linear molecules in good solvent, the radius of gyration is given by

$$R_G \approx N^{3/5} a \tag{4}$$

where N is the number of monomers in the coil and a is the monomer length.†

For a given combination of the polymer and the flow, one might expect that there

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† While some light scattering experiments (e.g. Virk 1975) show that $R \approx N^{1/2}a$, this difference appears to be a matter of definition. For example, Flory's 'length of chain link' refers to the simplest possible situation where each unit of the polymer is identical to the other, and is exactly situated at a distance a from the next. On the other hand, Virk's 'length of monomer' allows for the building blocks of polymers which are not single, identical, ideal 'points' but could be short chains themselves. For polyethyleneoxide (PEO), for example, a is the length of the backbone chain link of the short sequence given by

$$\mathrm{CH_2 - CH_2 - O},$$

and is not just that of a single, simple atom, monomer. So, we shall continue to use equation (4).

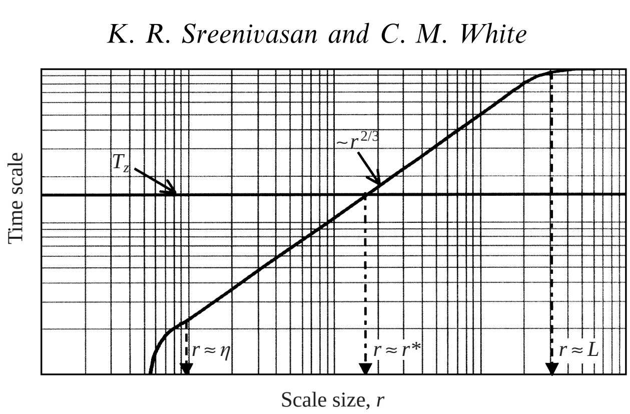

FIGURE 2. A schematic of the time scales in homogeneous turbulence and the definition of the length scale r^*. The two-thirds power law follows from Kolmogorov's (1941) phenomenology in high-Reynolds-number turbulence. This power law is approximately valid between some multiple of the Kolmogorov scale, η, and a fraction of the large scale, L.

exists a turbulent length scale r^* whose time scale τ_{r^*} matches T_z (see figure 2). This length scale is given by the relation

$$r^* = u_{r^*} T_z \tag{5}$$

where the velocity scale u_{r^*}, characteristic of the scale r^*, is given by Kolmogorov (1941) to be

$$u_{r^*} = (r^* \langle \varepsilon \rangle)^{1/3}. \tag{6}$$

Here and elsewhere, coefficients of order unity are omitted, mostly because we do not know them well enough. Polymer molecules can be expected to be stretched by all scales $r < r^*$. The elastic theory supposes that in a certain scale range $r^{**} < r < r^*$, the polymers are stretched little, and so advect without producing a reciprocal effect on the flow. That is, while the flow scales smaller than r^* do have an effect on the polymer, all flow scales larger than r^{**} remain unaffected by the polymer. The empirical basis of this assumption of no feedback has been discussed by de Gennes (1990, p. 38), and need not be repeated here; the principal point is that the stretching in this scale range is relatively small. The scale r^{**} is determined by the criterion that the elastic energy stored by the polymer molecules (per unit volume) equals the typical turbulent energy (per unit volume) at that scale (see figure 3). Turbulence scales finer than r^{**} are strongly affected by the elastic forces, the effect being the obstruction of the usual Kolmogorov-type energy cascade. It is not necessary for our purposes to speculate about the precise dynamics of the scales below r^{**}, as long as we can estimate its magnitude.

To continue, one needs to evaluate two quantities: (a) the amount by which the polymer molecule is stretched at any given scale $r < r^*$, and (b) the manner in which the elastic stress depends on the amount of stretching. For (a), the theory supposes that the stretching at any scale $r < r^*$ follows the power law

$$\lambda(r) = (r^*/r)^n, \tag{7}$$

where n depends on the dimensionality of stretching, equal to 1 in two dimensions and to 2 in three dimensions. For (b), it is argued by de Gennes that the elastic energy varies with $\lambda(r)$ as

$$G[\lambda(r)]^{5/2}, \tag{8}$$

where $G = c\kappa T/N$, c being the concentration of the polymer in units of monomers per

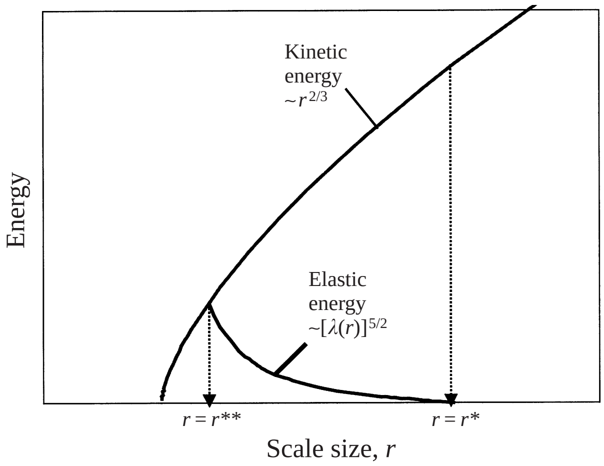

FIGURE 3. A schematic of the turbulent energy (per unit volume) scales, according to the Kolmogorov phenomenology, as $r^{2/3}$ (upper line), while the elastic energy (per unit volume) begins to grow at r^* according to $[\lambda(r)]^{5/2}$ (lower line). The scale at which the two quantities are equal defines r^{**}. The line representing the elastic energy has not been continued below r^{**} because the behaviour in this region is not understood.

unit volume, and all other symbols are defined earlier. The 5/2-power of λ, instead of the nominal power of 2, appears because of certain considerations of the repulsion of neighbouring polymer molecules (see Pincus 1976). It then follows that r^{**} is given by

$$G[\lambda(r^{**})]^{5/2} = \rho u_{r^{**}}{}^2, \tag{9}$$

the quantity on the right-hand side being the kinetic energy of turbulence at scale r^{**}. Equation (9) prescribes the dependence of r^{**} on the polymer concentration.

Now, for very small concentrations, the scale r^{**} will be smaller than η, and so the polymer will have no effect on turbulence. There exists a minimum concentration for which $r^{**} = \eta$, at which the polymer effects will begin to be just felt. This corresponds to the onset of drag reduction. Equation (9) thus shows how the molecular properties of the polymer enter the picture, and provides explicit predictions of the threshold concentration in terms of the polymer–flow combination. Note that, in contrast to much of the previous work, the molecular weight of the polymer does not appear anywhere so far; it is essentially subsumed by N, the number of monomers per coil.

As the concentration is increased further, a number of effects can arise. For example, the characteristic scale r^{**} may approach and equal the scale r^*, so that even a minute amount of stretching will render the elastic energy equal to the kinetic energy at this common scale. It is also possible that the increased concentration may induce neighbouring coils to interact, so that the turbulent rate-of-strain becomes an ineffective stretching mechanism. It is similarly possible that, in the case of an inhomogeneous flow, local strain rates may be so high at some places that the polymers are fully extended and elongational viscosity may indeed dominate. One has to sort through these various possibilities before concluding anything useful.

3.2. *Adapting the theory to pipe flow*

The elastic mechanism could in principle operate anywhere in the pipe, including its core region. If that happens, the drag reduction manifests itself because the core interacts with the wall region. Details of this interaction remain poorly understood despite intense efforts of the past (see e.g. Panton 1997), and the situation is not ripe for definitive analysis. However, we shall now consider the consequences of the elastic theory by *assuming* that turbulence at each radial position is essentially homogeneous, and that the physics discussed in § 3.1 for homogeneous turbulence is valid for each radial position. The inhomogeneity is felt through the fact that the scales r^* and r^{**} will now be functions of the radial position in the pipe – a situation which itself arises entirely through the radial variation of the energy dissipation, $\langle \varepsilon \rangle$. We shall assume the usual scaling for the energy dissipation: $\langle \varepsilon \rangle = u_\tau^4/\nu$ in the inner region, $\langle \varepsilon \rangle = u_\tau^3/R$ in the core region, and $\langle \varepsilon \rangle = u_\tau^3/y$ in the intermediate or overlap region.

With this information, the radial variations of r^* and r^{**} can be calculated according to equations (5), (6), (7) and (9). For example, for the intermediate region, it follows from equations (5) and (6) that

$$r^{*+} \equiv r^* u_\tau/\nu = De^{3/2}/y^{+1/2}, \tag{10}$$

where the suffix + indicates normalization by the wall variables u_τ and ν, and De is the so-called Deborah number, $De \equiv u_\tau^2 T_z/\nu$. From equations (6), (7) and (9), we obtain

$$r^{**+} \equiv r^{**} u_\tau/\nu = [(G/\rho u_\tau^2)(y^+/De)]^{\upsilon}(De^{3/2}/y^{+1/2}) \tag{11}$$

where the exponent υ, given by

$$\upsilon = (5n/2 + 2/3)^{-1}, \tag{12}$$

is not known *a priori* but depends, as already noted, on the dimensionality of stretching. The calculations of r^* and r^{**} are trivially similar for the wall and core regions.

The overall behaviours of r^* and r^{**} are shown schematically in figure 4 as functions of the radial distance from the pipe wall. Also shown for reference is the variation of the Kolmogorov scale, η. All quantities are normalized by the wall variables ν and u_τ. For the conditions shown, polymer action is different for different radial positions. Two typical behaviours are shown for positions marked y_1 and y_2. At y_1, the stretching occurs between the scales marked A_1 and B_1. Even when the polymer molecules continue to be stretched up to the Kolmogorov scale – which is as far down in scale as stretching can increase – the stored elastic energy remains smaller than the turbulent energy at any flow scale, and so polymers will have no effect on the flow. At y_2, the stretching occurs between A_2 and B_2 (i.e. r^{**} at y_2); at this point, the stored elastic energy equals the turbulent energy, and all scales at y_2 smaller than B_2 will be affected by the polymer.

4. The onset of drag reduction

In pipe flows of premixed polymer solutions, nothing special seems to happen in the viscous sublayer itself (e.g. Virk *et al.* 1967; Donohue, Tiederman & Reischman 1972). The rough-pipe studies of Virk (1971) confirm the conclusion. The conventional wisdom, which is quite old (Oldroyd 1948), is that important action occurs in the buffer region of the pipe flow, which is centred at $y^+ \approx 12$. This view is substantiated by studies in which polymer was injected at selected points in water flow (e.g. Wells & Spangler 1967; McComb & Rabie 1979): it was found that the polymer was most effective when it reached the buffer region.

For very low concentrations of the polymer, the intersection between the η^+ line and the r^{**+} line occurs (see figure 4) below the buffer region, and so the polymer

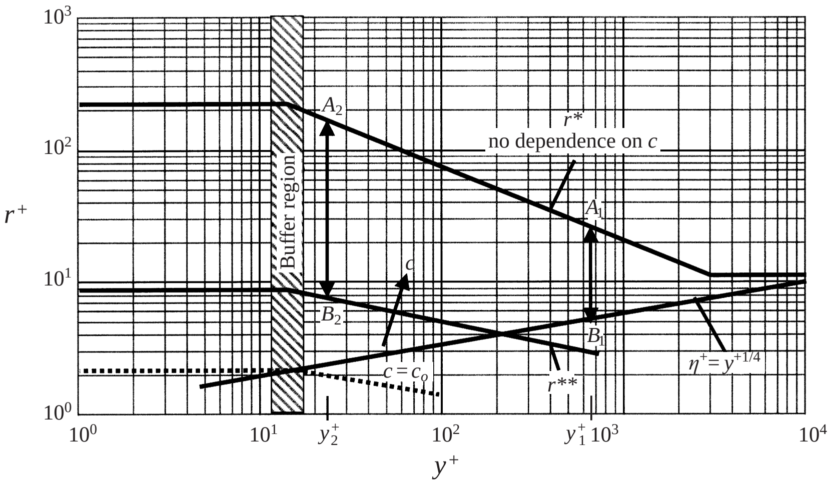

FIGURE 4. A schematic of the variation of the length scales r^* and r^{**} in a pipe flow. Also shown is the Kolmogorov scale. All scales are normalized by the wall variables, u_τ and ν. The dotted line for $c = c_o$ corresponds to the onset of drag reduction. See text for a discussion of other lines in the figure.

has no effect on any part of the flow. At some minimum concentration, say c_o, the intersection point occurs within the buffer region. This corresponds to the onset of drag reduction. In equation (11), this is equivalent to fixing y^+ at some value of the order 12, and equating r^{**} to η. This immediately yields

$$G_o/(\rho u_\tau^2) = De^{1-2/3v}, \tag{13}$$

G_o being the value of G when $c = c_o$. As already noted, v is still an unknown.

We examined a large body of experimental literature with a view to testing equation (13). While many observations exist on the onset of drag reduction, very few sources provide the details needed on the polymer characteristics such as T_z. Data from those sources have been plotted in figure 5 in the form suggested by equation (13). Each data set organizes roughly according to the power law (13) with about the same exponent. The coefficient of proportionality is not the same in each case – but perhaps we have no right to expect it to be so, given the limited accuracy to which the various quantities in equation (13) are known; for example, the relation between T_z and R_G might involve a numerical factor that depends on the distribution of the molecular weights of the monomers. Further, T_z might not be the only relevant relaxation time. However, if we were to ascribe a common slope to all the lines of figure 5, that slope would be roughly about $-5/2$, giving $v \approx 3/7$ (see equation (12)) and $n \approx 2/3$.

It has been noted already that the value of n is smaller if the stretching occurs in lower dimensions. The fact that $n = 2/3 < 1$ indicates, loosely speaking, that the effective stretching occurs in dimensions lower than 2. The fluctuating strain rates in the wall and buffer regions of the pipe are weaker in the radial and azimuthal directions than in the streamwise direction, and so most of the effective stretching there occurs in one-plus dimensions; this suggests consistency with the notion that the onset of drag reduction is principally a near-wall effect. The effect is presumed to be manifested through the interference with turbulent bursting near the wall, as seems to have been suggested first by Black (1968), but the causal connection

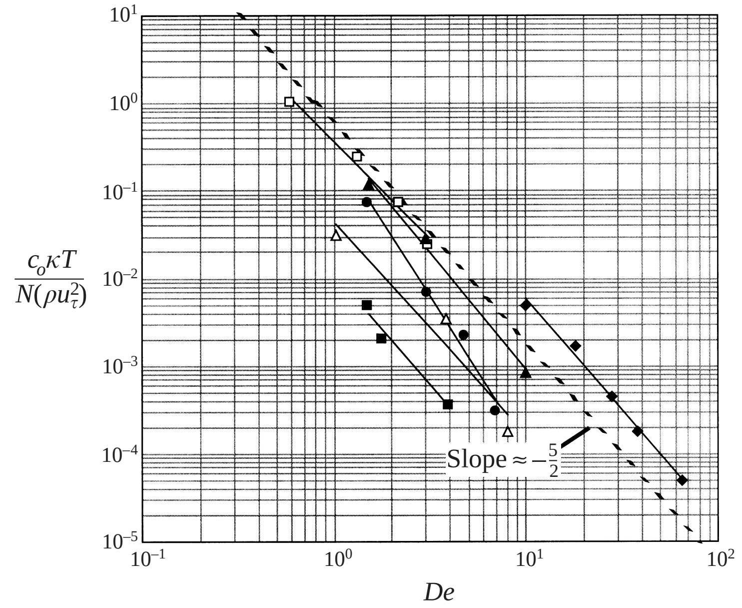

FIGURE 5. The available experimental data on the onset of drag reduction, as required by the elastic theory: ◆, ▲, △ from Nadolink (1987), $N = 6.1 \times 10^4$, 7×10^4, and 2.3×10^4, respectively; □, Berman (1977), $N = 1.4 \times 10^5$; ●, Patterson & Abernathy (1970), $N = 1.8 \times 10^5$; ■, present data, $N = 1.1 \times 10^5$. The dashed line has a slope of $-5/2$, which is close to the slopes of all the experimental data.

remains incompletely understood despite some imaginative work (see e.g. Goldshtik, Zametalin & Stern 1982).

The magnitude of n can be used to argue that the polymers are only partially stretched, thus lending support to one of the premises of the elastic theory. To see this, note that the maximum amount of stretching occurs when r^* and r^{**} are furthest apart. This magnitude is of the order $(R/\eta)^{2/3}$, R being the pipe radius. In turn, this is roughly proportional to $Re_\tau^{1/2}$ where $Re_\tau = u_\tau R/\nu = Re(f/2)^{1/2}$. For experiments discussed here, the maximum stretching factor turns out to be smaller than about 50. Given that polymer molecules have about 10^5 links in them, this calculation provides a self-consistency check on the assumption that the polymer molecules are most likely to remain poorly stretched at the onset.

5. The maximum drag reduction asymptote

The discussion so far suggests that the elastic theory may have the right elements in it, and may be useful also for extracting an explanation for the MDR asymptote. Such an explanation must possess some degree of universality because MDR is insensitive to most polymer–flow characteristics. In particular, explanations based on the bond breakage, which focus on polymer molecules, do not seem to be suitable.

The final arbiter has to be the agreement with experiment. Even though there are many sets of data that fall on the MDR asymptote (see e.g. Virk 1975), there are relatively few sets of experimental data just where the crossover to MDR occurs. We have examined all those latter data and tested various scenarios.

One scenario is that MDR occurs when the polymer concentration is high enough

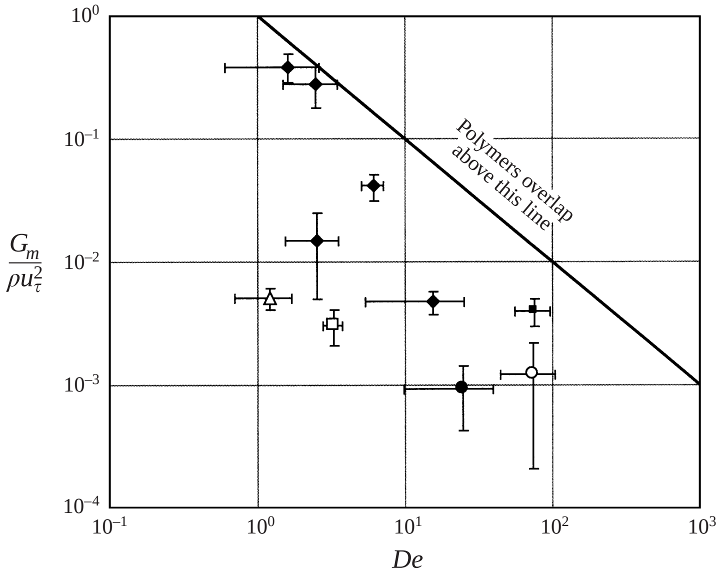

FIGURE 6. Log-log plot of $G_m/(\rho u_{\tau^2})$ vs. *De*. $G_m = c_m \kappa T/N$, c_m being the concentration as the drag reduction data approach the MDR asymptote. If the polymer coils are close enough to interact with each other, the data should lie on the straight line drawn. They fall substantially below the line, and suggest (despite the large uncertainty) that the MDR concentrations are smaller. The error bars reflect the uncertainty in reading the data off the graphs from primary sources (largely because the transition to the MDR asymptote is gradual), and do not necessarily reflect the ambiguity of experimental control. The sources are: △, present; □, Patterson & Abernathy (1970); ◆, Virk (1975); ●, Virk (1971); ○, Gampert & Wagner (1982); ■, Virk & Baher (1970).

that unstretched coils begin to overlap. This concentration is by definition of the order of N/R_G^3. The corresponding condition can be reduced, after noting (3), to

$$G_m/(\rho u_\tau^2) = De^{-1}, \tag{14}$$

where $G_m = c_m \kappa T/N$ and c_m is the concentration when the drag data just cross over to the MDR asymptote. Figure 6 plots $G_m/(\rho u_\tau^2)$ against *De*, and compares the data to equation (14). Almost all the data fall below the line (despite the large uncertainty), arguing that MDR occurs for concentrations below those required for the overlap of polymer coils. The argument is consistent with the empirical fact that MDR often occurs for concentrations that are only a few tens of parts per million by weight.

A related possibility is that the polymers are fully stretched at MDR, so that the elongational viscosity indeed becomes the controlling feature. Two factors argue against this possibility. First, simple calculations of maximum stretching – already mentioned – show that full stretching is unlikely. Second, MDR is independent of the polymer properties such as the number of monomers per chain, whereas the elongational viscosity depends on this latter quantity.

We shall therefore consider other possibilities. It is important to note that, while turbulence dynamics at the onset of drag reduction will not have changed from the standard state, substantial changes can be expected to occur when MDR sets in – and so something will have to be said about these changes. The flow at the onset of MDR does not relaminarize (Narasimha & Sreenivasan 1979), and velocity fluctuations reach a new self-sustaining state. Thus, the flow has to be treated as turbulent in some essential respects. The key feature of the elastic theory that allows us to make further progress is that it views the standard dynamics as prevailing for all scales larger than r^{**}, and that r^{**} is viewed effectively as the smallest turbulence scale.

To fix r^{**}, let us note that, in a standard turbulent flow, the energy dissipation can be expressed as a product of fluid viscosity and the square of the rate of strain at the smallest flow scale. Thus,

$$\langle \varepsilon \rangle = \nu (u_{r^{**}}/r^{**})^2, \tag{15}$$

where prefactors have been omitted as before, and $u_{r^{**}}$ is the velocity scale characteristic of r^{**}. Using the inertial-range relation (6) – which will be assumed, for consistency, to be valid for scales $r^* \geqslant r^{**}$ – equation (15) yields

$$u_{r^{**}} = \nu/r^{**}, \quad \text{and} \quad u_r^* = u_{r^{**}}(r^*/r^{**})^{1/3}. \tag{16}$$

From equation (5), we also obtain

$$r^*/r^{**} = [\nu T_z/r^{**2}]^{3/2}. \tag{17}$$

Combining this relation with equations (7) and (9), we finally have

$$\frac{G_m}{\rho u_\tau^2} De^{15n/4} = \left[\frac{u_\tau r^{**}}{\nu}\right]^{15n/2-2}. \tag{18}$$

Taking $n = 2/3$ from the previous section, we may rewrite equation (18) as

$$\left[\frac{G_m}{\rho u_\tau^2}\right]^{1/3} De^{5/6} = \frac{u_\tau r^{**}}{\nu}. \tag{19}$$

This result must hold when MDR is approached. It is, however, not testable because the right-hand side has never been measured. Writing the Reynolds number $u_\tau r^{**}/\nu$ as a power of the friction Reynolds number, Re_τ^β, where β is an unknown index to be determined, the above equation becomes

$$\left[\frac{G_m}{\rho u_\tau^2}\right]^{1/3} De^{5/6} = Re_\tau^\beta. \tag{20}$$

This equation is testable on the basis of available evidence.

The strategy now is to plot the two sides of equation (20) for different values of β, and pick the one that yields the least error. We have found that $\beta \approx 1$ yields the smallest error. The corresponding results are shown in figure 7: as required, a plausible linear relation exists between $[G_m/(\rho u_\tau^2)]^{1/3} De^{5/6}$ and Re_τ.

The result that $\beta \approx 1$ implies that r^{**} near the MDR asymptote scales with the pipe radius, R. It is difficult to say precisely how much of a fraction of R is r^{**} because of the lack of control on various prefactors, but we suspect that it is small. It is easy to see from equation (17) that r^*/r^{**} would be $O(De^{3/2}/Re_\tau^3)$. This ratio is of the same order in all the experiments analysed here, suggesting that r^* too is of the order of the pipe radius. The interpretation is that, at the MDR asymptote, even the small amount of stretching that is characteristic of length scales of order R will immediately render the elastic energy of polymers comparable to turbulent energy. This, according to the present analysis, is the basic physics of MDR.

6. Concluding remarks

The addition of small amounts of long-chain polymers to flowing fluids produces many interesting changes. At the present level of understanding, most of those effects

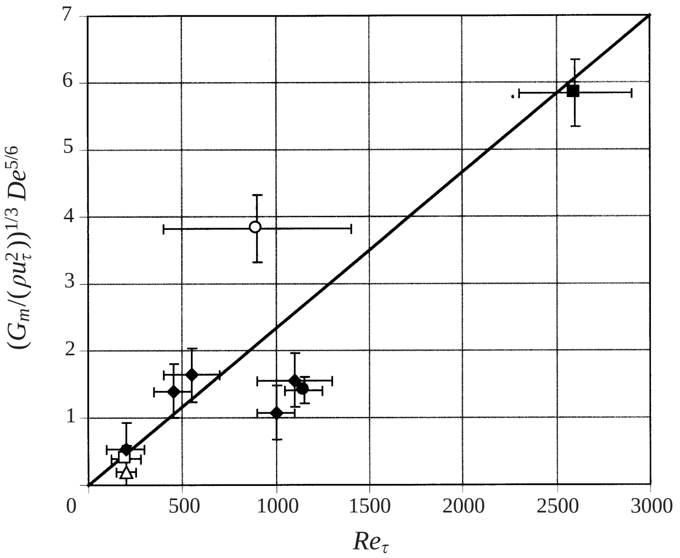

FIGURE 7. Linear plot of $[G_m/(\rho u_\tau^2)]^{1/3}De^{5/6}$ as a function of the friction Reynolds number, Re_τ. The present theory requires the data to plot on a straight line. This seems plausible. The error bars and symbols have the same meanings as in figure 6.

cannot be formulated in terms of a theory. Two aspects that can be formulated are the onset and the approach to the MDR asymptote. The theory in question is the elastic theory of de Gennes and Tabor. Recent experiments of Smith & Chu (1998) suggest that one of the premises of the theory, namely that the polymer coils are not stretched fully in the presence of large fluctuating strains, is probably correct. This provides an incentive for exploring the consequences of the theory. By patching up the theory in some respects, we have extracted relations that are testable experimentally, and compared them with available observations in pipe flows. These comparisons have not been made before. The evidence on the onset problem indicates that the theory is not inconsistent with experimental data. In particular, in the case of premixed polymer solutions, the action that determines the onset of drag reduction appears to occur near the wall, not in the core region. The conclusion on MDR is less reliable. However, the suggestion is that, near MDR, even the weakest rates of strain, characteristic of the large scale, will render the polymer's elastic energy comparable to the turbulent kinetic energy.

The present results in terms of the elastic mechanism need not imply that those in terms of extensional viscosity are incorrect. Any reasonable working model of polymers (such as the Finitely Extensible Nonlinear Elastic, or FENE) suggests that an elastic effect can formally be interpreted in terms of a corresponding viscous effect, despite fundamental differences between the two mechanisms. It is thus premature, in our opinion, to close the door on any specific mechanism; accordingly, our own conclusions are to be treated as tentative.

This tentativeness arises for at least two additional reasons: (i) the behaviour of turbulence in the presence of polymer is still largely unknown, and various theoretical assumptions are required to cover this ignorance – for instance, about the manner in which polymers stretch in the presence of randomly varying, non-Gaussian, fluctuating rates of strain; (ii) the data used to verify the theory are themselves tentative at least because of the absence of serious characterization of the polymer. It may thus seem that we have here a situation in which an incomplete theory has been compared with inadequate data. Unfortunately, this is nearly all that can be done at present. To improve the situation, one requires new experiments at high Reynolds numbers, an integral part of which should be the characterization of the polymer. Such experiments must be directed towards problems that have a chance of being posed adequately in theoretical terms – such as the ones chosen here – rather than to the exploration of a whole range of fascinating but ill-posed issues. High Reynolds numbers are required because – if we understand anything at all in turbulence – it is its behaviour in the presence of large scale-separation. Meaningful interaction between the theory and experiment may then be possible, leading to fuller understanding.

Professor Philip Saffman has expanded fluid mechanics on many fronts and left an indelible mark on the subject. We are pleased to dedicate this article to him. We thank Dr Richard Nadolink and Professors Robert Powell and Pierre-Gilles de Gennes for their interest, and to Professor John Hinch and Dr John Rallison for a brief but instructive tutorial. The work was supported by the US National Science Foundation, grant number CTS-9520155.

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